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COMMENT

Comment on an exact solution of the coupled nonlinear Schrödinger equations

D Kapor, M Škrinjar and S Stojanović

Institute of Physics, Faculty of Sciences, 21000 Novi Sad, Yugoslavia

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Abstract. We show that a previously presented solution for the system of coupled nonlinear differential equations leads to inconsistent results except in the case where the system is decoupled.

Recently Lan and Wang (LW) have published papers [1, 2] on possible methods for obtaining exact solutions of some nonlinear differential equations. Here we shall concentrate on the coupled nonlinear Schrödinger equations which arise in the study of monomode step-index optical fibres [3]. Our aim is to show that the solutions presented by LW [2] are in fact inconsistent, and cannot be used.

We study the following system:

$$\mathrm{i}\frac{\partial A^+}{\partial \tau}=\frac{\partial^2 A^+}{\partial x^2}+A^+(|A^+|^2+\hbar|A^-|^2) \tag{1}$$

$$\mathrm{i}\frac{\partial A^-}{\partial \tau}=\frac{\partial^2 A^-}{\partial x^2}+A^-(|A^-|^2+\hbar|A^+|^2). \tag{2}$$

Here A^+ and A^- are the amplitudes of the electric field and $\hbar$ is a parameter [3]. The first step is the transition to new variables:

$$A^+(x)=A_1(x)\exp(\mathrm{i}\rho_1\tau) \tag{3}$$

$$A^-(x)=A_2(x)\exp(\mathrm{i}\rho_2\tau) \tag{4}$$

where ρ_1 and ρ_2 are two parameters which LW obviously assume to be real (otherwise, one could not obtain the following set of equations). These equations are

$$\frac{\partial^2 A_1}{\partial x^2}+\rho_1 A_1+A_1(|A_1|^2+\hbar|A_2|^2)=0 \tag{5}$$

$$\frac{\partial^2 A_2}{\partial x^2}+\rho_2 A_2+A_2(|A_2|^2+\hbar|A_1|^2)=0. \tag{6}$$

The next step is the ansatz:

$$A_1(x)=a\tanh\mu x \tag{7}$$

$$A_2(x)=b\,\mathrm{sech}\,\mu x. \tag{8}$$

Here a, b and μ are the parameters to be determined. Combining equations (5)–(8), LW obtain the following set:

$$-2a\mu^2+\rho_1 a+\hbar ab^2=0 \tag{9}$$

$$-2a\mu^2-a^3+\hbar ab^2=0 \tag{10}$$

$$b\mu^2+\rho_2 b+\hbar ba^2=0 \tag{11}$$

$$-2b\mu^2+b^3-\hbar ba^2=0 \tag{12}$$

and they obtain

$$a^2=-\rho_1 \tag{13}$$

$$b^2=(2\rho_2-\rho_1)/(\hbar-2) \tag{14}$$

$$\mu^2=\rho_1(1+\hbar)/2 \tag{15}$$

and the constraint

$$\rho_1=2\rho_2/(\hbar-1). \tag{16}$$

In fact, (13) is obtained directly from (9) and (10). But from (9) and (12), using (13), one arrives at

$$b^2=\rho_1 \qquad (\hbar\neq 1). \tag{17}$$

So, there appears an inconsistency: the set (9)–(12) is obtained under the assumption that both a and b are real, but (13) and (17) show that one of them has to be purely imaginary, if ρ_1 is real. (One can arrive at (17) also by combining (14), (15) and (16).)

Therefore, let us assume for the moment that a and b can be complex numbers. The set (9)–(12) now becomes:

$$-2a\mu^2+\rho_1 a+\hbar a|b|^2=0 \tag{18}$$

$$-2a\mu^2-a|a|^2+\hbar a|b|^2=0 \tag{19}$$

$$b\mu^2+\rho_2 b+\hbar b|a|^2=0 \tag{20}$$

$$-2b\mu^2+b|b|^2-\hbar b|a|^2=0. \tag{21}$$

It follows from (18) and (19) that

$$|a|^2=-\rho_1 \tag{22}$$

and introducing (22) into (21) gives

$$-2\mu^2+|b|^2+\hbar\rho_1=0 \tag{23}$$

which, combined with (18), gives

$$|b|^2=\rho_1. \tag{24}$$

Once again, we arrive at an inconsistency. We see that, contrary to the statement of Lan and Wang, as long as a and b are non-vanishing, or $\hbar\neq 1$, a and b cannot be chosen in a consistent manner, so the discussion based on the set (13)–(16) that follows in (2) is not relevant at all.

Let us mention that the system (9)–(12) allows decoupled solutions: $a=0$, $|b|^2=2\mu^2=-\rho_2^2$ or $|a|^2=-2\mu^2=-\rho_1$, $b=0$. These solutions correspond to circularly polarized modes already discussed in [3].

The most interesting case is $\hbar = 1$. According to Shadevan *et al* [4], this corresponds to one of the only two possible parameter choices for the general case of coupled NSE which posseses the Painlevé property so it is completely integrable [5]. In this case (18) and (19) lead again to (22), but (21) becomes equal to (18), so we conclude that:

$$|b|^2 = 2\mu^2 - \rho_1 = \rho_1 - 2\rho_2 \tag{25}$$

and

$$\mu^2 = \rho_1 = \rho_2. \tag{26}$$

We see that the solutions depend on two parameters ρ_1 and ρ_2.

We must now look for the physical implication of the formal problem. According to Newboult *et al* [3] parameter $\hbar = f_3/f_2$ is defined as the ratio of two positive quantities given by the integrals of the squares of amplitudes of field components. They are related by:

$$f_3 = 2f_2 - 32\omega\varepsilon_0 n_q \int_0^\infty n_j E_2^2 E_3^2 r \,\mathrm{d}r$$

where all quantities are positive. The condition that both f_2 and f_3 are positive implies that $f_3 - f_2 < -f_2$ so it can never vanish, f_3 can never be equal to f_2 and $\hbar$ is never equal to 1.

We see now that the combination of mathematical and physical constraints prevents us from using the solutions of the system (1) and (2) in the form (3)–(8), except in the decoupled case, when this is trivial.

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